Practical criterion for delay estimation using random perturbations

M. Siefert*
Institute of Physics, University of Potsdam, Potsdam, Germany
(Received 15 February 2007; published 23 August 2007)

Many systems contain an internal time delay, which significantly influences their dynamical properties. Methods to estimate this delay from times series in the presence of dynamical noise are not systematically studied. Addressing this problem, we demonstrate that it is sufficient to analyze the system's response to short-correlated external disturbances or internal noise. Following this idea, it is shown for linear and nonlinear systems, as well as for periodic dynamics, that the delay can be estimated by analyzing the correlation function. This method covers the case of strong noise and multiple delays.

DOI: 10.1103/PhysRevE.76.026215 PACS number(s): 05.45.Tp, 02.30.Ks, 02.50.Fz, 05.40.−a

I. INTRODUCTION

Understanding of the dynamical behavior of complex real systems is a fundamental problem in physics. A variety of measurement techniques are available for observing the dynamics of the systems with high temporal resolution. Further insight into the underlying physical process governing the dynamics can be obtained through time series data analysis, which helps to extract the most important dynamical features even if one does not have enough understanding of its components. The ability of recovering the features of the dynamics heavily depends on model assumptions and methodology of data analysis. A key feature of real systems is the presence of time delays in interaction between system components. Time delayed dynamics plays an increasingly important role in various fields of science including biology [1–3], chemistry [4,5], climatology [6,7], (nonlinear) optics [8,9], and transport systems [10,11] among others. Time delays can occur over a broad range, with time scales spanning, for example, from nanoseconds in laser systems up to several hours in genetic networks. In general, time delays lead to a very complicated and infinite high dimensional dynamics. The identification and quantification of time delays from time series is therefore an important task.

There exist different methods for reconstructing the delay from given time series. Many of them rely on an extension of the embedding theorem for deterministic chaotic systems and uses the fact that the number of variables needed to embed the system is drastically reduced if the delay is chosen correctly [12–18]. They differ mainly in the way this property is tested. A related approach is to estimate numerically a model from the time series [19–21] or to estimate the unknown parameters for such a model [22]. The forecast error of these models is minimized if the correct delay is chosen.

Most of the above methods do not take into account dynamical noise, i.e., noise which directly acts on the system dynamics, even though they may work for small noise levels. Furthermore, these methods can fail for nearly periodic states [23]. Methods which are explicitly based on *stochastic* models and therefore consider dynamical noise are not systematically studied in the literature. Linear models have been considered by [24], coupled oscillators with a delayed coupling have been considered by [23], where dynamical noise was essential for delay estimation.

In this paper we follow the idea that the delay could be estimated by studying the response of the system to perturbations of the system. The response is partially delayed and identifying the general features of the response could help to estimate the time delay. We will show that even the response of the system to intrinsic noise, which is present in many systems, is sufficient to estimate the delay and we introduce a practical criterion for this.

To be more specific, let us consider the mathematical description of time delayed systems. Because of the delay, the system's dynamics cannot be described by an ordinary differential equation (ODE) $\dot{x}(t)=f(x(t))$, where the time evolution is only determinate by the present state. Additionally the state $x(t-\tau)$ at time τ ago has to be taken into consideration, where τ is the time delay. Therefore, such systems can be more realistically modeled with a delayed differential equation (DDE)

$$\dot{x}(t) = f(x(t),x(t-\tau)). \tag{1}$$

Due to the time delayed feedback, the initial condition has to be a function $\phi(t)$, $t\in[0,\tau]$, the prefunction, and the dynamics is much more complex in comparison to an ODE. For an introduction to DDEs, see, for example, [25–27].

If the system involves uptake and dissipation of energy, as is usual in the case of complex systems, fluctuations of the dynamics are present. This can be approximated mathematically by a *stochastic* delay differential equation (SDDE),

$$\dot{x}(t) = f(x(t),x(t-\tau)) + \eta(t), \tag{2}$$

where x is the actual state of the system, $\eta(t)$ is a stochastic force with some distribution, some correlation function, and with vanishing mean $\langle\eta(t)\rangle=0$. This SDDE is the underlying model of the analysis presented in this paper.

In this paper we focus on linear and nonlinear stochastic delay differential equations also including the case of multiple delays. A *sufficient* criterion is given, from which the delay's magnitude can be estimated. The noise plays the role of a disturbance of the system and by analyzing the system's response to it, the delay can be estimated.

*malte.siefert@gmail.com

The paper is structured as follows. First, we present the idea of the method and analytically the operability of the proposed method for a linear SDDE. Then we extend the underlying idea to nonlinear systems by means of Gaussian approximation. The results are confirmed by analyzing numerical generated time series. Finally, we present the estimation of multiple delays.

II. THEORY OF THE DELAY ESTIMATION

A. Linear dynamics

In the following, the linear case is considered. It can be analyzed exactly and explain how the delay can be estimated by means of the responses of the system to the noise. Due to the delay the response is partially retarded and by examining the interaction between noise and delay the typical fingerprint of the time delay in the correlation function can be found.

The starting point is the linear stochastic delay differential equation (SDDE)

$$\frac{\partial x(t)}{\partial t} = ax(t) + bx(t-\tau) + \eta(t). \tag{3}$$

Here $\eta(t)$ denotes noise, which is allowed to have a finite, but short correlation time. By multiplying this equation with x and after averaging, one gets an equation for the correlation $C_{xx}(t)=\langle x(s)x(s+t)\rangle$. Similarly, an equation for the cross-correlation function $C_{\eta x}(t)=\langle \eta(s)x(s+t)\rangle$ results by multiplying Eq. (3) by the noise η:

$$\frac{\partial}{\partial t}C_{xx}(t) = aC_{xx}(t) + bC_{xx}(t-\tau) + C_{x\eta}(t), \tag{4}$$

$$\frac{\partial}{\partial t}C_{\eta x}(t) = aC_{\eta x}(t) + bC_{\eta x}(t-\tau) + C_{\eta\eta}(t). \tag{5}$$

The correlation $C_{\eta\eta}$ is given by the existing noise. Using the symmetry $C_{x\eta}(t)=C_{\eta x}(-t)$, which is valid for stationary processes, one can see that Eqs. (4) and (5) are closed equations for the correlation functions. Assuming that the noise has a short correlation time $t_{\mathrm{corr}} \ll \tau$, the cross-correlation function $C_{x\eta}(t)$ can be obtained as follows (see Fig. 1). For $t < -t_{\mathrm{corr}}$ the noise does not influence the state x and accordingly the cross correlation vanishes, $C_{\eta x}(t)=0$. This means the prefunction for Eq. (5) is zero and we have the initial condition for Eq. (5). Approaching $t=0$, $C_{\eta\eta}$ becomes finite and $C_{\eta x}(t)$ increases fast due to the small width of $C_{\eta\eta}(t)$; the increase happens within times of the order of t_{corr}. For $0<t<\tau$ the solution decays exponentially with time rate a. Figure 1 shows a typical sketch of this behavior for $C_{\eta x}(t)$.

If noise actually helps to estimate the delay, then the response of $x(t)$ around $t\approx\tau$ to the noise should reveal the delay. Therefore, we establish in the following a relation between C_{xx} around $t\approx\tau$ and the noise containing term $C_{x\eta}(t)$ around $t\approx 0$.

Equation (4) is differentiated once and in the second term on the right hand side the time reversal symmetry of the correlation function is used:

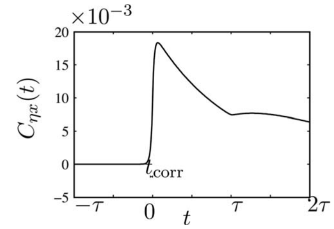

FIG. 1. The typical course of the cross correlation function $C_{\eta x}(t)$ [see Eq. (5)] with $t_{\mathrm{corr}}=\tau/50$. Due to causal reasons, for $t < -t_{\mathrm{corr}}$ the correlation is zero. Within $-t_{\mathrm{corr}}<t<t_{\mathrm{corr}}$ the correlation function $C_{\eta\eta}$ of the noise has its maximum and $C_{\eta x}(t)$ increases steeply. For $t>t_{\mathrm{corr}}$ the noise has no influence anymore and the system responds with an exponential decay, followed for $t>\tau$ by an evolution, which depends on the parameters a and b.

$$\frac{\partial^2}{\partial t^2}C_{xx}(t) = a\frac{\partial}{\partial t}C_{xx}(t) - b\frac{\partial}{\partial u}C_{xx}(u)\bigg|_{u=\tau-t} + \frac{\partial}{\partial t}C_{x\eta}(t). \tag{6}$$

Inserting again Eq. (4) in this equation it follows that

$$\begin{aligned}\frac{\partial^2}{\partial t^2}C_{xx}(t) &= a[aC_{xx}(t) + bC_{xx}(t-\tau) + C_{\eta x}(-t)] - b[aC_{xx}(\tau \\ &\quad - t) + bC_{xx}(-t) + C_{\eta x}(t-\tau)] + \frac{\partial}{\partial t}C_{\eta x}(-t) = (a^2 \\ &\quad - b^2)C_{xx}(t) + aC_{\eta x}(-t) - bC_{\eta x}(t-\tau) + \frac{\partial}{\partial t}C_{\eta x}(-t)\end{aligned} \tag{7}$$

and since the correlation time of the noise is smaller than the delay, it is for $t\approx\tau$:

$$\frac{\partial^2}{\partial t^2}C_{xx}(t) \approx (a^2-b^2)C_{xx}(t) - bC_{\eta x}(t-\tau). \tag{8}$$

This equation relates the response of the correlation function C_{xx} in the vicinity of $t=\tau$ to the noise's influence around $t=0$. In this equation a remarkable property becomes visible. Due to the dependence on the cross correlation, the second derivative $\frac{\partial^2}{\partial t^2}C_{xx}$ changes fast around $t=\tau$, cf. Fig. 1. If the noise becomes δ correlated, the fast change becomes a jump in the second derivate. This observation will be used in the following as an estimator for the delay time.

B. Nonlinear dynamics

In the following nonlinear systems are considered and it will be shown that the delay can be estimated analogous to the linear case by analyzing the correlation function. We use the Gaussian approximation to demonstrate that the method works for nonlinear systems. The starting point is the nonlinear stochastic delay equation

$$\frac{\partial}{\partial t}x(t) = f(x(t),x(t-\tau)) + \eta(t). \tag{9}$$

Similar to the linear case, $\eta(t)$ denotes noise, which can be correlated as long as the correlation time is much shorter than the delay time. After multiplying this equation by $x(t)$ and $\eta(t)$, respectively, and by subsequent averaging, we get a differential equation for $C_{xx}(t)$ and for $C_{\eta x}(t)$, respectively. On the right hand side, terms of the form $\langle xf\rangle$ or $\langle \eta f\rangle$ occur. Using the Gaussian approximation, we can use the Furutsu-Novikov equation:

$$\langle x_k f[x_1,x_2]\rangle = \langle x_k x_j\rangle \left\langle \frac{\partial f[x_1,x_2]}{\partial x_j}\right\rangle. \tag{10}$$

This approximation is strictly valid for Gaussian distributed processes also if they are arbitrary nonlinear. With it we end up with the two equations

$$\frac{\partial}{\partial t}C_{xx}(t) = C_{xx}(t)\langle \partial_1 f[x(0),x(-\tau)]\rangle + C_{xx}(t-\tau)\langle \partial_2 f[x(0),x(-\tau)]\rangle + C_{x\eta}(t) \tag{11}$$

and

$$\frac{\partial}{\partial t}C_{\eta x}(t) = C_{\eta x}(t)\langle \partial_1 f[x(0),x(-\tau)]\rangle + C_{\eta x}(t-\tau)\langle \partial_2 f[x(0),x(-\tau)]\rangle + C_{\eta\eta}(t), \tag{12}$$

where we have used the stationarity of the process. ∂_i denotes the derivate with respect to the function's ith argument. Both equations are similar to the linear case (4) and (5). The second derivate is given by

$$\frac{\partial^2}{\partial t^2}C_{xx}(t) = (\langle f_1\rangle^2 - \langle f_2\rangle^2)C_{xx}(t) + \langle f_1\rangle C_{x\eta}(t) - \langle f_2\rangle C_{\eta x}(t-\tau) + \frac{\partial}{\partial t}C_{x\eta}(t), \tag{13}$$

where the abbreviation $\langle f_i\rangle = \langle \partial_i f[x(0),x(-\tau)]\rangle$ is used. For $t \approx \tau$ it is

$$\frac{\partial^2}{\partial t^2}C_{xx}(t) \approx (\langle f_1\rangle^2 - \langle f_2\rangle^2)C_{xx}(t) - \langle f_2\rangle C_{x\eta}(\tau - t). \tag{14}$$

For short correlation times of the noise, the delay can again be identified by a jump in the second derivative of the correlation function, which is a manifestation of the system's response after time τ to the short disturbances of the system.

In the following, numerically generated time series of three different systems are analyzed to verify the idea of the delay estimation. The first example is a linear system, the second is a periodic limit cycle, and the third one is a nonlinear system with two delays.

III. NUMERICAL EXAMPLES

A. Linear case

As a first example, the correlation function of the linear system (3) with parameter values $a=-1$, $b=-1$, $\tau=1$, and δ-correlated noise is addressed. For $0 \le t \le \tau$ the solution is $C_{xx}(t) = C_0 - \frac{D}{2}t$ with the diffusion constant $D \equiv \int_{-\infty}^{\infty}\langle \eta(0)\eta(t)\rangle dt$ [28]. For $\tau \le t \le 2\tau$ this solution can serve as the prefunction and $\partial_t C_{xx} = aC_{xx} + b\left[C_0 - \frac{D}{2}(t-\tau)\right]$ gives the solution of the second interval, and so on ("method of steps"). Between the two intervals, the second derivative is discontinuous. We solve the SDDE numerically using a Heun scheme to show how the delay can be estimated from a time series [29]. The diffusion coefficient is $D=0.01$, the step size is $\Delta t = 10^{-3}$. From the time series we calculate the correlation function using $N=10^6$ data points with a Savitzky-Golay filter (comparable results are also obtained with a simple second order difference). The parameters of this differentiation has to be varied in order to give the best results.

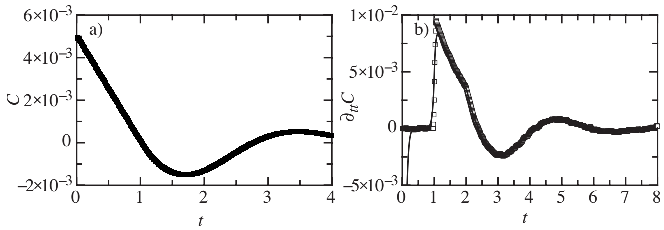

FIG. 2. (a) The correlation function for the linear SDDE (3) with $a=-1$, $b=-1$, $D=0.01$ calculated from the numerical solution of the SDDE with step size $\Delta t = 0.001$ from $N=10^6$ data points. The case of correlated noise is not distinguishable from the δ-correlated case. (b) Second derivative of the correlation function for δ-correlated noise (symbols) as well as for noise with finite correlation length of $\tau/20$ (line).

Figure 2 shows the result for δ-correlated noise as well as for noise with finite correlation length $\tau/20$. The correlated noise is generated by an Ornstein-Uhlenbeck process. The second derivative clearly shows a jump at the delay time; see Fig. 2(b).

B. Nonlinear dynamics

We present two examples to show that the delay produces a distinct signature in the correlation also in the case of nonlinear systems. The first example is the Mackey-Glass equation. We chose the periodic regime for which it is not possible to estimate the delay without external perturbation or noise, respectively. The second example, a laser equation, considers the case of multiple delays.

C. Periodic case

The following system is a model for the production of white blood cells proposed by Mackey and Glass [2]:

$$\dot{x} = -x + \epsilon \frac{x_\tau}{1 + x_\tau^n}. \tag{15}$$

Here we chose $\epsilon=2$ and $n=10$ for which the solution lies on a limit cycle. For this case it is not possible to distinguish the dynamics from an ordinary differential equation and thus it is not possible to estimate the delay. This can be explained as follows: The nature of a delay equation is that a function ϕ defined on the interval $[t-\tau, t]$ can unambiguously describe the system's time evolution. In the present case, due to the periodicity, the knowledge of *one* point $(x(t-\tau'), x(t))$ is sufficient to estimate the future. But this is also the property of

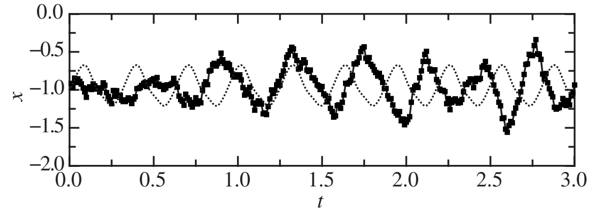

FIG. 3. Time series of the Mackey-Glass equation in the periodic regime, with noise $D=0.05$ (symbols) and without noise (line). The parameters are $\epsilon=2$, $n=10$, $\tau=1$.

an ordinary differential equation. Because dissipative systems typically contract to low dimensional manifolds, there exists a broad class of systems, where the estimation of the delay is not possible without disturbances or noise.

We disturb the system with a Langevin force of strength $D=0.05$. Figure 3 shows the numerically generated time series of the undisturbed as well as of the disturbed system. From $N=10^6$ data points the correlation function is calculated; see Fig. 4. The second derivative of the correlation function clearly shows the discontinuity at the delay time; see Fig. 4(b). Thus after disturbances are applied to the system, the delay can be estimated from the time series.

IV. ESTIMATION OF MULTIPLE DELAYS

In certain situations, more than one delay governs the dynamics. An example is a laser with two feedbacks, where the light takes two different ways back to the laser. For many circumstances the Ikeda equation is a model for laser systems with feedback [8]. We extend the model for wavelength dynamics [30] to the case of two delays and take into account dynamical noise:

$$\dot{x}(t) = -x(t) - \{\sin^2[x(t-\tau_1)] + \sin^2[x(t-\tau_2)]\} + \eta(t). \tag{16}$$

The delays $\tau_1=10$ and $\tau_2=13$ are chosen. The system is disturbed by noise with strength $D=0.01$. The length of the time series is $t=10\,000$, from which the correlation function is calculated; see Fig. 5(a). The second derivative of the correlation function clearly shows the two delays in the form of two sharp bends; see Fig. 5(b). Notice that the bends are exactly located at the delay times and are not displaced due to the nonlinearity [31,32]. Similar results are obtained for arbitrary different delay times.

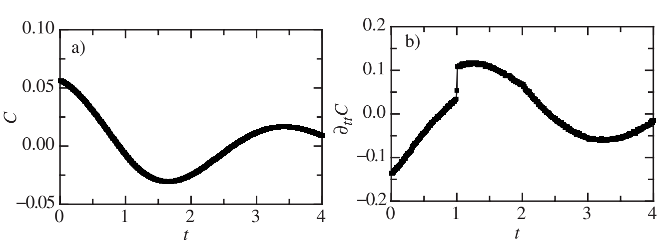

FIG. 4. The correlation function of the Mackey-Glass equation calculated from the time series with $N=10^6$ data points and noise strength $D=0.05$. Parameters are the same as in Fig. 3.

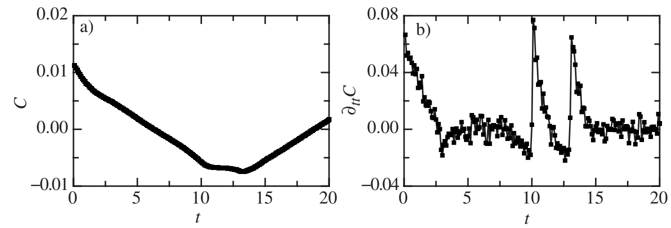

FIG. 5. The correlation function of the Ikeda-model (16) with two delays at $\tau_1=10$ and $\tau_2=13$ disturbed by noise with strength $D=0.01$. (a) The correlation function and (b) the second derivative of the correlation function.

V. CONCLUSION

The idea of this paper is to use inevitable system inherent or external random perturbations to identify the delay from time series. It has been shown that by using these perturbations, it is possible to estimate the delay by a discontinuity in the correlation function. This can be shown rigorously for linear systems; see Eq. (8). For a nonlinear system a Gaussian approximation has been used to get an analogous result to the linear case; see Eq. (14). The only condition is that the noise is short-correlated compared to the delay time and the strength is sufficiently large. The results have been tested for different numerical examples. Even for a noisy periodic limit cycle of the highly nonlinear Mackey-Glass system, the delay can be estimated. Furthermore, also multiple delays can be identified as it is shown for the Ikeda model with two feedbacks. This idea closes some gaps for delay estimation presented up to now in the literature. Most work was done to identify the delay from chaotic systems with—if any—moderate dynamical noise [12–24,31,33]. For stochastic delay differential equations the results in the literature are not very systematic. One method exists for linear systems [24], and another for coupled oscillators where the delay is in the coupling [23]. All the methods have in common that it is necessary to decorrelate the dynamics between $x(t)$ and $x(t-\tau)$ either by chaos or by noise. In this sense the method presented here is an evident extension for noisy systems: it is based solely on the response of the system to the noise or, in other words, on the interaction of the delay with the noise. Thus it uses the constructive role of noise and therefore relies on sufficient high noise amplitudes. The use of the presented approach for multivariate variables is straightforward. The only required precondition is that the noise acts directly on the observed variable. A potential limitation of the procedure is the calculation of the second derivative of the correlation function. This demands a statistic which converts fast enough and a time series which is relatively long; but the length of the time series is of comparable size as in different approaches; see, for example, [33]. The other crucial limitation is that large enough noise amplitudes are needed. A condition for sufficient strong noise is that the correlation function drops linearly at the origin. If, however, the noise amplitude is too small, the discontinuity in the second derivative is less pronounced. In this case the deterministic dynamics dominates the dynamics and the other cited methods may be used with better results. To conclude, for systems with internal or external short-correlated perturbations, it is sufficient to analyze the correlation function to estimate the delay, which is very simple to perform.

I acknowledge helpful suggestions from Arkady Pikovsky and helpful discussions with Michael Rosenblum. This project has been supported by the DFG (SFB 555 "Complex Nonlinear Processes").

[1] A. Longtin, J. G. Milton, J. E. Bos, and M. C. Mackey, Phys. Rev. A **41**, 6992 (1990).
[2] M. C. Mackey and L. Glass, Science **1977**, 287 (2006).
[3] V. Volterra, J. Math. Pures Appl. **7**, 249 (1928).
[4] I. R. Epstein, Int. Rev. Phys. Chem. **11**, 135 (1992).
[5] M. R. Roussel, J. Phys. Chem. **100**, 8323 (1996).
[6] A. J. Clarke, X. Liu, and S. Van Gorder, J. Clim. **11**, 987 (1998).
[7] E. Tziperman, L. Stone, M. A. Cane, and H. Jarosh, Science **264**, 72 (1994).
[8] K. Ikeda, Opt. Commun. **30**, 257 (1979).
[9] R. Lang and K. Kobayashi, IEEE J. Quantum Electron. **16**, 347 (1980).
[10] M. Bando, K. Hasebe, K. Nakanishi, and A. Nakayama, Phys. Rev. E **58**, 5429 (1998).
[11] K. Nakanishi, Phys. Rev. E **62**, 3349 (2000).
[12] M. J. Bunner, M. Ciofini, A. Giaquinta, R. Hegger, H. Kantz, R. Meucci, and A. Politi, Eur. Phys. J. D **10**, 165 (2000).
[13] M. J. Bunner, M. Ciofini, A. Giaquinta, R. Hegger, H. Kantz, R. Meucci, and A. Politi, Eur. Phys. J. D **10**, 177 (2000).
[14] M. J. Bunner, T. Meyer, A. Kittel, and J. Parisi, Phys. Rev. E **56**, 5083 (1997).
[15] A. C. Fowler and G. Kember, Phys. Lett. A **175**, 402 (1993).
[16] R. Hegger, M. J. Bunner, H. Kantz, and A. Giaquinta, Phys. Rev. Lett. **81**, 558 (1998).
[17] D. T. Kaplan and L. Glass, Physica D **64**, 431 (1993).
[18] Y. C. Tian and F. R. Gao, Physica D **108**, 113 (1997).
[19] B. P. Bezruchko, A. S. Karavaev, V. I. Ponomarenko, and M. D. Prokhorov, Phys. Rev. E **64**, 056216 (2001).
[20] S. P. Ellner, B. E. Kendall, S. N. Wood, E. McCauley, and C. J. Briggs, Physica D **110**, 182 (1997).
[21] H. Voss and J. Kurths, Phys. Lett. A **234**, 336 (1997).
[22] W. Horbelt, J. Timmer, and H. U. Voss, Phys. Lett. A **299**, 513 (2002).
[23] L. Cimponeriu, M. Rosenblum, and A. Pikovsky, Phys. Rev. E **70**, 046213 (2004).
[24] T. Ohira and R. Sawatari, Phys. Rev. E **55**, R2077 (1997).
[25] R. Bellmann and K. L. Cooke, *Differential-Difference Equations* (Academic Press, New York, 1963).
[26] O. Diekmann, *Delay Equations: Functional-, Complex-, and Nonlinear Analysis* (Springer-Verlag, New York, 1995).
[27] J. K. Hale, *Theory of Functional Differential Equations* (Springer-Verlag, New York, 1977).
[28] T. D. Frank, P. J. Beek, and R. Friedrich, Phys. Rev. E **68**, 021912 (2003).
[29] J. Honerkamp, *Stochastic Dynamical Systems. Concepts, Numerical Methods, Data Analysis* (Wiley-VCH, New York, 1993).
[30] C. Marriott, R. Vallee, and C. Delisle, Phys. Rev. A **40**, 3420 (1989).
[31] M. J. Bunner, A. Kittel, J. Parisi, I. Fischer, and W. Elsasser, Europhys. Lett. **42**, 353 (1998).
[32] S. Lepri, G. Giacomelli, A. Politi, and F. T. Arecchi, Physica D **70**, 235 (1994).
[33] M. J. Bunner, M. Popp, T. Meyer, A. Kittel, U. Rau, and J. Parisi, Phys. Lett. A **211**, 345 (1996).